ELSEVIER

Journal of Hydrology 175 (1996) 407–428

Journal of Hydrology

An analysis of the dynamic component of the geomorphologic instantaneous unit hydrograph

Marco Franchini[a,*], P. Enda O'Connell[b]

[a] *Università di Bologna, Facoltà di Ingegneria, Istituto di Costruzioni Idrauliche, Viale Risorgimento 2, 40136 Bologna, Italy*

[b] *Department of Civil Engineering, University of Newcastle upon Tyne, Newcastle upon Tyne NE1 7RU, UK*

Received 2 March 1995; accepted 30 March 1995

Abstract

A geomorphologic instantaneous unit hydrograph (IUH) consists of two components, one relevant to the geomorphology and the other to the hydraulic aspect describing the movement of a drop of water along a stream. Different formulations of the geomorphologic IUH are reviewed, and a contrast is drawn between the geomorphologic and hydraulic components of the geomorphologic IUH (GIUH) proposed by Rodriguez-Iturbe and Valdés (*Water Resour. Res.*, 15(6); 1409–1420, 1979) and those of a width function based IUH (WFIUH). In this paper a comparison has been carried out of the original GIUH and a WFIUH which allows the effects of different geomorphologic and hydraulic components to be identified. The comparison, which is based on four sub-basins of the River Tyne, UK, clearly shows that the GIUH velocity parameter lacks physical interpretation, in contrast to the hydraulic parameters of the WFIUH, which are seen to be physically consistent. For practical application of the GIUH, an equation is then proposed to estimate the velocity parameter through the basin concentration time, the Horton length ratio and the length of the stream of the highest order of the channel network.

1. Introduction

Recently, many attempts have been made to relate the response of a catchment to its morphologic or topologic aspects, using various hypotheses to model both the advection and attenuation effects of a river network. In a significant development, Rodriguez-Iturbe and Valdés (1979) introduced the concept of the geomorphologic instantaneous unit hydrograph (GIUH), later generalized by Gupta et al. (1980). The

* Corresponding author.

SSDI 0022-1694(95)02852-8

basic idea of the GIUH is that the distribution of arrival times at the basin outlet of a unit instantaneous impulse injected throughout a channel network is affected both by the underlying natural order in the morphology of the catchment and the hydraulic characteristics of the flow along the channels themselves. In the original approach of Rodriguez-Iturbe and Valdés (1979), the underlying natural order in the morphology is represented by the Horton ratios which, in turn, are based on a classification of the channel network of the catchment according to Strahler's ordering scheme, whereas the holding time of a drop of water within a stream of a given order is represented by means of an exponential law which is, however, a conceptualization of the true flow dynamics. As a consequence of this last hypothesis, the average holding time of a drop within a stream of a given order is proportional to the average length of all the streams of that order, and the proportionality factor is the velocity of the water, which is considered uniform throughout the drainage basin (this assumption has been known and often used in many hydrological models since the studies of Leopold and Maddock (1953) and Pilgrim (1976, 1977)).

Again examining the channel network response, Rinaldo et al. (1991) basically used the same conceptual structure as used by Rodriguez-Iturbe and Valdés (1979), i.e. they also referred to a Strahler ordered basin, but they replaced the assumption of exponential holding times with the inverse Gaussian distribution, which, in turn, is the impulse response of the convective diffusion equation representing the flow in a stream of the network.

Other workers (Kirkby, 1976; Mesa and Mifflin, 1986: Naden, 1992) have proposed different formulations of the geomorphologic IUH based on the width function (WF) of the basin coupled with various routing procedures (these formulations are denoted here as WFIUHs). In particular, the observed WF of a basin is coupled with the convective diffusion equation in the case of the Mesa and Mifflin (1986), and Naden (1992) WFIUH formulations. In these cases, the hydraulic component is characterized by two parameters which represent the celerity and the longitudinal diffusivity. These parameters are dependent on the geomorphic characteristics of local slope and discharge, implying that at least the order of magnitude of these quantities is physically determined. Finally, Troutman and Karlinger (1984, 1985, 1986) and Karlinger and Troutman (1985) proposed the topologic IUH, which is based, unlike a WFIUH, on the specification of a finite number of topologic features rather than the complete WF.

Because, in each geomorphologic IUH, it is always possible to identify two components, as mentioned above, i.e. one relevant to the geomorphology and the other relevant to the hydraulic aspect describing the movement of a drop of water along a stream, comparison between the formulations can be made, hydraulic and/or geomorphologic conditions being equal. On the basis of this idea, Snell and Sivapalan (1994) compared the geomorphologic IUH proposed by Rinaldo et al. (1991) with the Mesa and Mifflin (1986) approach. The geomorphologic IUH proposed by Rinaldo et al. (1991) was parametrized (1) by using the Horton order ratios to derive analytical expressions for the geomorphologic parameters and (2) by extracting these directly from a Strahler ordered network without using the Horton order ratios. The scope of that paper was to analyse different approaches by which geomorphology can be introduced through the probabilities and lengths of the pathways available within the network, hydraulic descriptions being equal.

As the original GIUH of Rodriguez-Iturbe and Valdés (1979) is in widespread use today, a comparison is also needed between this formulation and the WFIUH approach, to identify clearly the effects of both different geomorphologic and hydraulics components. In this paper, Naden's WFIUH approach (1992) is considered. Such a comparison has been performed by considering four sub-basins of the River Tyne, from which the geomorphologic information was extracted, and by defining common hydraulic conditions for both formulations, i.e. the velocity for the GIUH and the celerity and the diffusivity for the WFIUH. The analysis of these real-world cases clearly shows the physically meaningless nature of the GIUH velocity parameter in contrast to the hydraulic parameters of the WFIUH, which are seen to be physically consistent. For practical application of the GIUH, an equation is then proposed to estimate the velocity parameter through the basin concentration time, the Horton length ratio and the length of the stream of the highest order of the channel network.

Finally, it is worth noting that there is an underlying assumption throughout this paper of negligible travel time spent by a drop along the hillslopes. Therefore, the geomorphologic IUHs considered here are strictly related to the channel network. This assumption may be questionable within the framework of the overall representation of the rainfall–runoff process at basin level; nevertheless, it is used here simply to focus attention on the representation of channel network response alone.

2. The geomorphologic instantaneous unit hydrograph (GIUH)

The GIUH proposed by Rodriguez-Iturbe and Valdés (1979) is based on Shreve's theory (1966) of topologically random networks of a given magnitude (i.e. a given number of sources) and on the state-transition approach (Howard, 1971) coupled with a Markov process. In this formulation, the state i identifies the location of an individual drop of water within a stream of order i or in the area drained by a stream of order i; in a drainage network, a transition can only occur from a given state i to some state of higher order j; the probability of that transition is defined as

$$p_{ij} = \frac{\text{number of streams of order } i \text{ draining into order } j}{\text{number of streams of order } i} \tag{1}$$

and the probability of state i is given by

$$\theta_i = \frac{\text{total area draining directly into the streams of order } i}{\text{total area of the basin}} \tag{2}$$

To derive a distribution for the travel time to the outlet of an individual particle, it is necessary to hypothesize a holding time distribution for each state of the system. Rodriguez-Iturbe and Valdés (1979) assumed that the probability density function of the time spent by a generic drop in a state of order i is

$$f_i(t) = \lambda_i \exp(-\lambda_i t) \tag{3}$$

where λ_i is the reciprocal of the mean holding time in any stream of order i. This position is equivalent to treating each order of stream as a linear reservoir.

The application of Eq. (3) to all orders of stream including the highest would imply a hydrograph for the whole basin which does not start from zero. To avoid this, Rodriguez-Iturbe and Valdés (1979) split the highest-order stream into two streams (i.e. two linear reservoirs) in series, each with a travel time probability distribution

$$f_\Omega(t) = \lambda_\Omega^* \exp(-\lambda_\Omega^* t) \tag{4}$$

where $\lambda_\Omega^* = 2\lambda_\Omega$, i.e. each with a mean holding time of $0.5\lambda_\Omega^{-1}$. According to Rodriguez-Iturbe and Valdés (1979), only the first, from upstream, of the two 'reservoirs' that represent the highest-order stream, receives (in the case, for example, of a third order channel network) the drops from all second-order streams, a proportion of the first-order streams, and those drops draining directly into the third-order stream.

Using Eqs. (1)–(3) and Howard's state-transition theory coupled with the Markov process, the probability distribution of the total travel time to the outlet, i.e. the GIUH, can be derived. However, this approach is complicated analytically and may not be strictly necessary for deriving the GIUH. Gupta et al. (1980) restated the basic concepts, and derived the cumulative density function of the time of travel to the basin outlet as

$$P(T_B \leqslant t) = \sum_{s \subset S} P(T_s \leqslant t) P(s)$$

where $P(\cdot)$ stands for the probability of the set given in parentheses, T_B is the travel time to the basin outlet, T_s is the travel time for a particular path s, $P(s)$ is the probability of a drop taking paths s, and S is the set of all possible paths that a drop can take upon falling in the basin.

Furthermore, Chutha and Dooge (1990) have shown that, for a given basin ordered according to Shreve's theory, the deterministic concept of routing through linear reservoirs (Nash, 1957; Dooge, 1959, 1973) defines an IUH which is identical to the GIUH. It is interesting to reproduce here such a derivation for a basin of third order by using this latter approach, because it provides a good insight into the lumping nature of the GIUH, and helps considerably in interpreting the results of the numerical examples described below.

Let us consider a typical third-order basin, represented in Fig. 1(a), where it is possible to recognize five types of sub-basin, as described in the figure caption. On the other hand, Fig. 1(b) represents the schematization of the basin used as the basis of the GIUH formulation (compare it with Fig. 3 of Rodriguez-Iturbe and Valdés (1979)). Fig. 1(b), in turn, can be used as a reference scheme to apply the convolution from the streams of first order up to the outlet of the basin. It should be noted that all the streams of a given order are lumped together, thus losing their real position and distance from the outlet.

In accordance with the previously introduced assumption, and, in particular, recalling that in the GIUH approach the drops of water falling in the area drained by streams of order i are instantaneously transferred to the corresponding stream of order i (no overland flow occurs), we consider an impulse of excess rainfall $\delta(\tau)$

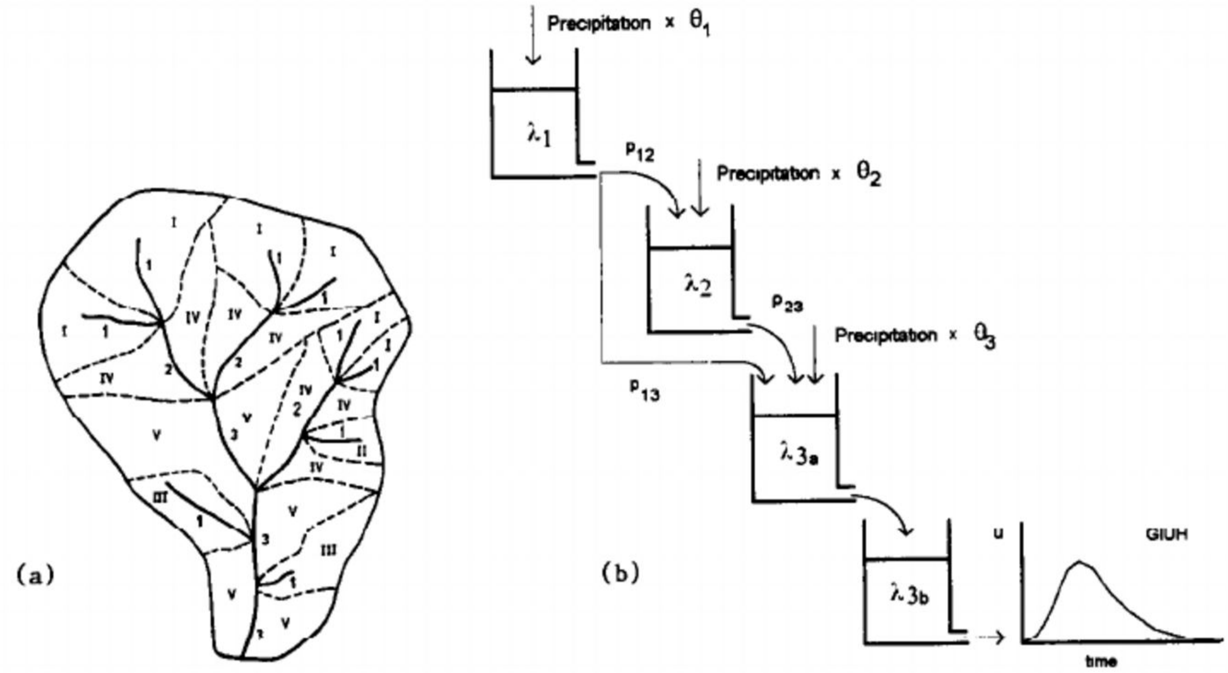

Fig. 1. A typical third-order basin and representation as combinations of unequal linear reservoirs. (a) Typical third-order basin with types of sub-basins. I, First-order stream forming a second-order stream; II, first-order stream tributary to second-order stream; III, first-order stream tributary to third-order stream; IV, overland flow to second-order stream; V, overland flow to third-order stream. (After Chutha and Dooge (1990).) (b) Representation of a third-order basin as a combination of linear storage elements in parallel and in series. Every reservoir represents all the streams of a given order. θ_i represents the probability that a raindrop falls within the area of the basin drained by a stream of order i, and p_{ij} represents the transition probability.

uniformly distributed over the whole channel network. Thus, the impulse contribution to the second-order streams coming from the first-order streams is (see Fig. 1(b)):

$$u_{12}(t) = \int_0^t [\theta_1 \delta(\tau)] p_{12} \lambda_1 \exp[-\lambda_1 (t - \tau)] \mathrm{d}\tau$$

and the impulse contribution to the first of the two reservoirs relating to the third-order streams (Reservoir a) is

$$u_{13_a}(t) = \int_0^t [\theta_1 \delta(\tau)] p_{13} \lambda_1 \exp[-\lambda_1 (t - \tau)] \mathrm{d}\tau$$

Similarly, the impulse contribution to the third-order reservoirs a coming from the reservoir relevant to the second-order streams, is

$$u_{23_a}(t) = \int_0^t [\theta_2 \delta(\tau)] \lambda_2 \exp[-\lambda_2 (t - \tau)] \mathrm{d}\tau + \int_0^t u_{12}(\tau) \lambda_2 \exp[-\lambda_2 (t - \tau)] \mathrm{d}\tau$$

and the impulse contribution to the third-order reservoir b is

$$u_{3_a 3_b}(t) = \int_0^t [\theta_3 \delta(\tau)] \lambda_3^* \exp[-\lambda_3^*(t-\tau)] \mathrm{d}\tau + \int_0^t u_{23_a}(\tau) \lambda_3^* \exp[-\lambda_3^*(t-\tau)] \mathrm{d}\tau$$
$$+ \int_0^t u_{13_a}(\tau) \lambda_3^* \exp[-\lambda_3^*(t-\tau)] \mathrm{d}\tau$$

Finally, the output from the 'reservoir' 3_b, i.e. the GIUH of the whole basin, is

$$\mathrm{GIUH} = u(t) = \int_0^t u_{3_a 3_b}(\tau) \lambda_3^* \exp[-\lambda_3^*(t-\tau)] \mathrm{d}\tau$$
$$= a \exp[-\lambda_1 t] + b \exp[-\lambda_2 t] + [ct - (a+b)] \exp[-\lambda_3^* t] \quad (5)$$

where

$$a = \theta_1 \lambda_1 (\lambda_3^*)^2 \left[p_{13} + \frac{p_{12}\lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3^* - \lambda_1)^2} \right]$$

$$b = \lambda_2 (\lambda_3^*)^2 \left[\theta_2 - \frac{\theta_1 p_{12} \lambda_1}{(\lambda_2 - \lambda_1)(\lambda_3^* - \lambda_2)^2} \right]$$

$$c = (\lambda_3^*)^2 \left\{ \theta_3 - \frac{\theta_1 p_{13} \lambda_1}{(\lambda_3^* - \lambda_1)} - \left[\theta_2 \lambda_2 - \frac{\theta_1 p_{12} \lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3^* - \lambda_2)} \right] - \frac{\theta_1 p_{12} \lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)(\lambda_3^* - \lambda_1)} \right\}$$

$$\lambda_3^* = 2\lambda_3$$

To summarize, the geomorphologic information embedded in the GIUH is represented by: (1) θ_i, the probability of state i; (2) p_{ij}, the transition probability from a given state i to some state of higher order j; (3) the way in which the reservoirs are combined, where each is representative of all the streams of a given order. It should be noted that this combination is equivalent to an arrangement of linear reservoirs in parallel and in series (Bras, 1990; Chutha and Dooge, 1990).

Furthermore, the parameter λ_i represents a size or scale characteristic of the basin, and, generally, the number of λ_i is equal to the order of the basin. To embed in this parameter the 'dynamic component' of the response of the basin, Rodriguez-Iturbe and Valdés (1979) defined an average velocity v for the catchment. Then

$$\lambda_i = v/\overline{L}_i \quad (6)$$

where $\overline{L}_i$ is the average length of the streams of order i. This hypothesis is based on the assumption that, for a given rainfall–runoff event, the streamflow velocity is approximately the same at any moment in time throughout the basin (Leopold and Maddock, 1953; Pilgrim, 1977). Furthermore, this 'velocity' can be taken as the velocity at the peak discharge time for a given storm event (Rodriguez-Iturbe et al., 1979). Finally, it should be noted that this hypothesis means that the average travel time spent by a drop travelling along a stream of a given order is proportional to its average length.

In the case of a pure 'Hortonian basin' (i.e. a basin where the Horton laws hold exactly), the probability of state i, θ_i, the transition probability p_{ij} and the parameters λ_i may be expressed in terms of the Horton ratios (see Appendix), to obtain the following results for a third-order basin (Rodriguez-Iturbe and Valdés, 1979):

$$\theta_1 = R_B^2 R_A^{-2} \tag{7a}$$

$$\theta_2 = \frac{R_B}{R_A} - \frac{R_B^3 + 2R_B^2 - 2R_B}{R_A^2(2R_B - 1)} \tag{7b}$$

$$\theta_3 = 1 - \frac{R_B}{R_A} - \frac{1}{R_A^2}\left[\frac{R_B(R_B^2 - 3R_B + 2)}{(2R_B - 1)}\right] \tag{7c}$$

$$p_{12} = \frac{R_B^2 + 2R_B - 2}{(2R_B^2 - R_B)} \tag{7d}$$

$$p_{13} = \frac{R_B^2 - 3R_B + 2}{(2R_B^2 - R_B)} \tag{7e}$$

$$\lambda_1 = v/\overline{L}_1, \qquad \lambda_1 = \lambda_1/R_L, \qquad \lambda_3 = \lambda_2/R_L^{-2} \tag{7f}$$

However, these equations, given the fact that they refer to the ensemble properties of basins, are expected to produce results which differ from the corresponding quantities estimated directly for any particular basin. Furthermore, small negative values of θ_i may result, particularly for higher orders. Bras (1990) simply suggested 'adjusting' those values to 'eliminate the aberrant behaviour'.

Hereafter the GIUH calculated through Eqs. (7a)–(7f) will be referred to as 'Horton GIUH', whereas, in the case where the quantities p_{ij}, θ_i and λ_i are directly calculated through Eqs. (1), (2) and (6), respectively, the GIUH will be referred to as 'actual GIUH'.

3. The width function instantaneous unit hydrograph (WFIUH)

Generally, a WFIUH is the combination of the WF with any possible linear routing scheme. In the present paper the routing scheme adopted is that proposed by Naden (1992).

3.1. The network width function

The width function gives a description of the network of a basin simply by using the distances from the outlet, measured along all its streams (Kirkby, 1976; Mesa and Mifflin, 1986). Its derivation is then simple: it is sufficient to plot the number of streams at successive distances away from the basin outlet as measured along the network itself. Fig. 2 shows the derivation of the WF. Under the assumption of uniform drainage density and constant flow velocity, the WF is equivalent to the time–area plot (Baldwin and Potter, 1987). Of course, for the calculation of the WFIUH, the WF is normalized to one by division by the total number of unit distance segments represented in the function. Furthermore, it can be easily coupled with any routing scheme, possibly of hydraulic type such as the convective diffusion equation, so that the values of the parameters can be defined in a sensible way, in accordance with the hydraulic characteristics of the channel network.

Finally, even if the following aspect is not considered in this paper, it is worth noting that many basin characteristics, such as the soil types (Naden, 1992), the slopes of the different branches of the river network, the drainage density (i.e. the basin area drained per unit length of the river network, etc.), can be easily embedded in the transfer component through the WF, by simply subjecting the WF to different weight sets, each of them relevant to a particular aspect of the basin. The effect of these weights consists in a modification of the WF itself, which should reflect more accurately the geomorphologic structure of the basin network.

3.2. The routing component

The use of the WF with the simple assumption of constant velocity easily converts its distance axis into a time axis, thus producing a discretized unit hydrograph. However, this approach disregards any possible attenuation effect owing to the storage capacity of the many streams in the network. For this reason, Naden (1992) suggested a fuller routing procedure based on the solution of the convective-diffusion equation:

$$\frac{\partial Q}{\partial t} = D\frac{\partial^2 Q}{\partial x^2} - c\frac{\partial Q}{\partial x} \tag{8}$$

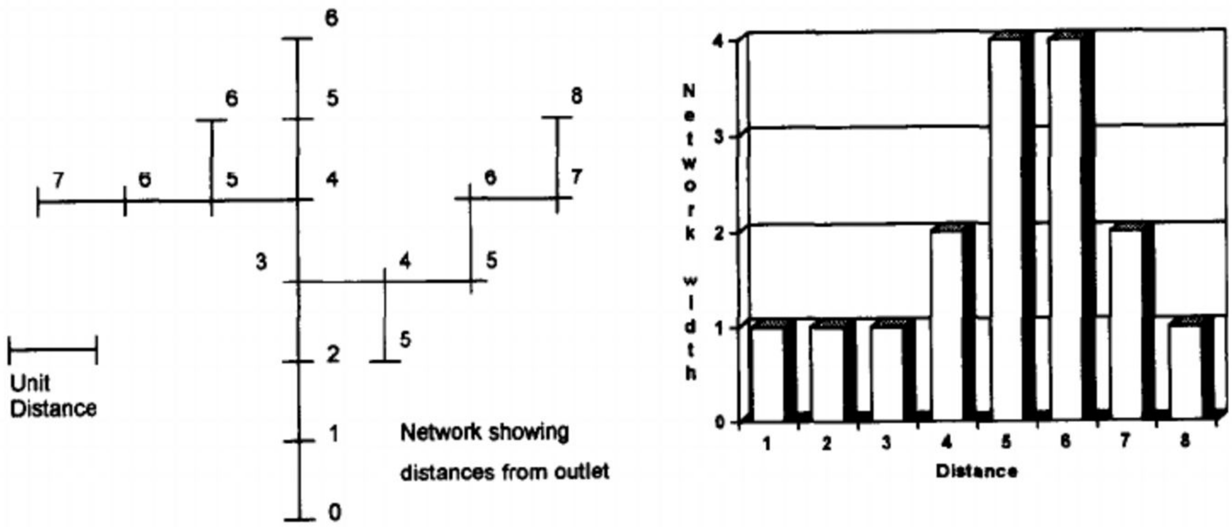

Fig. 2. Derivation of the network width function.

with boundary conditions $Q(0,t) = \delta(t)$, $Q(x,0) = 0$ and $Q(\infty, t) = 0$, where Q is discharge ($m^3\ s^{-1}$), D is diffusion coefficient ($m^2\ s^{-1}$), c is kinematic celerity ($m\ s^{-1}$), t is time (s) and x is distance from upstream end (m).

When the coefficients D and c are considered constant, the solution to Eq. (8) with the prescribed boundary conditions, is

$$h(x,t) = \frac{x}{2(\pi D t^3)^{1/2}} \exp\left[-\frac{(x-ct)^2}{4Dt}\right] \tag{9}$$

where $h(x,t)$ represents the impulse response of the convective diffusion equation, i.e. the time evolution of the discharge at a distance x from the upstream end when an instantaneous upstream impulse $\delta(t)$ is introduced.

From Eq. (9) it is easy to obtain the solution of Eq. (8) when a uniformly distributed lateral impulse enters the channel over a reach length $S = [(x = S) - (x = 0)]$. This is

$$\begin{aligned} h^l(S,t) &= \int_0^S \frac{x}{2(\pi D t^3)^{1/2}} \exp\left[-\frac{(x-ct)^2}{4Dt}\right] \mathrm{d}x \\ &= \frac{D^{1/2}}{(\pi t)^{1/2}} \left\{ \exp\left[-\frac{(ct)^2}{4Dt}\right] - \exp\left[-\frac{(S-ct)^2}{4Dt}\right] \right\} \\ &\quad + \frac{c}{2} \left\{ \operatorname{erf}\left[\frac{ct}{2(Dt)^{1/2}}\right] - \operatorname{erf}\left[\frac{(S-ct)^2}{2(Dt)^{1/2}}\right] \right\} \end{aligned} \tag{10}$$

where erf(·) represents the error function (Abramowitz and Stegun, 1965).

Finally, the solution at distance S of Eq. (8), when a uniformly distributed lateral impulse enters the channel over a reach length $\Delta x = [(x = x^*) - (x = 0)]$ with $x^* < S$, may be obtained directly from Eq. (10):

$$h^l_{\Delta x}(S,t) = \frac{1}{\Delta x}[h^l(S,t) - h^l(S - \Delta x, t)] \tag{11}$$

3.3. *The WFIUH of the whole basin*

Let Δx denote the length of the unit-distance segment used to build the WF; then $S = n\Delta x$ is the distance from the outlet of the basin of the upstream border of each segment. The WFIUH of the overall basin is then

$$\mathrm{WFIUH}(t) = \sum_{n=1}^{m} \mathrm{WF}_n \frac{1}{\Delta x} \{h^l(n\Delta x, t) - h^l[(n-1)\Delta x, t)]\} \tag{12}$$

where WF_n is the WF value for the nth set of streams and m is the most remote set of streams in the network.

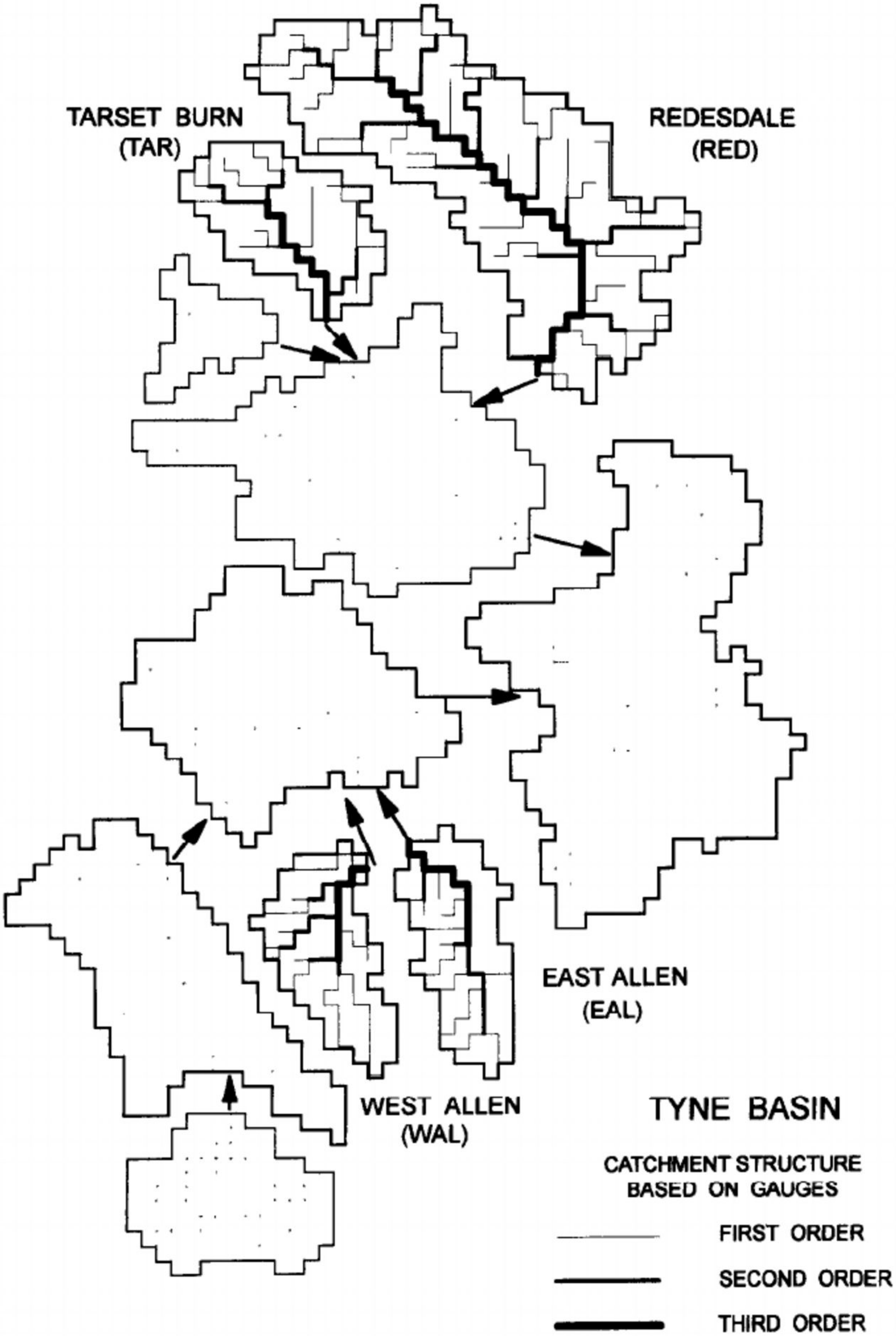

Fig. 3. Simplified grid-based drainage structure of the Tyne basin and the four sub-basins selected: TAR, RED, EAL and WAL. All the data used in the calculations were derived from the simplified representation of the Tyne basin.

4. A numerical comparison of the two models

The models described above have been compared using the geomorphologic information derived from four sub-basins of the River Tyne (UK) and then defining a common hydraulic condition for both of them, i.e. by defining v for the GIUH and c and D for the WFIUH in a consistent way.

The four basins used for the analysis of the two models are schematically represented in Fig. 3. They are the Tarset Burn (TAR) sub-basin, the Redesdale (RED) sub-basin, the East Allen (EAL) sub-basin, and the West Allen (WAL) sub-basin. The corresponding Horton ratios have been calculated by using the simplified grid-based drainage structure shown in Fig. 3 (grid size 1 km) and are presented in the first three columns of Table 1 together with the estimates of the probability of state i, θ_i, and of the transition probabilities p_{ij}, both obtained by using the formulae (7a)–(7e). The corresponding actual values, derived using Eqs. (1) and (2), are also given. The data which are necessary to calculate the actual values of θ_i and p_{ij} were extracted from the above-mentioned simplified map of the River Tyne. The comparison of the estimated with the actual values gives an idea of the differences that may arise when the two estimation methods are applied. Finally, the average lengths of the streams of given order are presented in the last three columns of Table 1.

In Fig. 4, the WFs of the four sub-basins are given. In these plots the y-axis represents the frequency, and the x-axis represents the distances from the outlet expressed in kilometres. Obviously, the maximum value on the x-axis is the distance of the farthest point of the river network from the basin outlet.

The common hydraulic conditions for the comparison have been established on the basis of the following assumptions. A wide rectangular cross-section has been assumed at the basin outlet; then, realistic values of width, slope, depth and roughness have been defined. This allows for the calculation of the velocity of the water v, celerity c and diffusion D; these hydraulic conditions can be thought of as relevant to the peak phase of a flood event. Finally, these values have been considered representative for the whole basin. This is consistent with the basic assumption in the

Table 1
Geomorphologic parameters of the four third-order sub-basins of the River Tyne

Basin code	R_A	R_B	R_L	Hortonian values					Actual values					$\bar{L}_1$ (km)	$\bar{L}_2$ (km)	$L_{\Omega=3}$ (km)
				θ_1	θ_2	θ_3	p_{12}	p_{13}	θ_1	θ_2	θ_3	p_{12}	p_{13}			
TAR	4.7	3.5	2.1	0.60	0.28	0.12	0.82	0.18	0.59	0.24	0.17	0.92	0.08	2.8	3.5	12
RED	8.2	5.9	3.4	0.52	0.36	0.12	0.70	0.30	0.47	0.29	0.24	0.78	0.22	3.1	4.7	37
EAL	5.0	3.5	2.1	0.51	0.30	0.19	0.82	0.18	0.40	0.26	0.34	0.75	0.25	2.7	3.3	11
WAL	4.1	3.3	1.5	0.78	0.26	−0.04	0.80	0.20	0.67	0.13	0.20	0.82	0.18	3.7	3.0	8

The Hortonian values of θ_i and p_{ij} were calculated using Eqs. (7). The 'actual values' of θ_i and p_{ij} were calculated by using Eqs. (1) and (2), respectively, and the data necessary for the application of these last two equations were extracted directly from the simplified grid-based drainage structure of the Tyne basin. $\bar{L}_1$ and $\bar{L}_2$ represent the average length of the streams of Order 1 and 2, respectively, and $L_{\Omega=3}$ represents the length of the stream of the highest order.

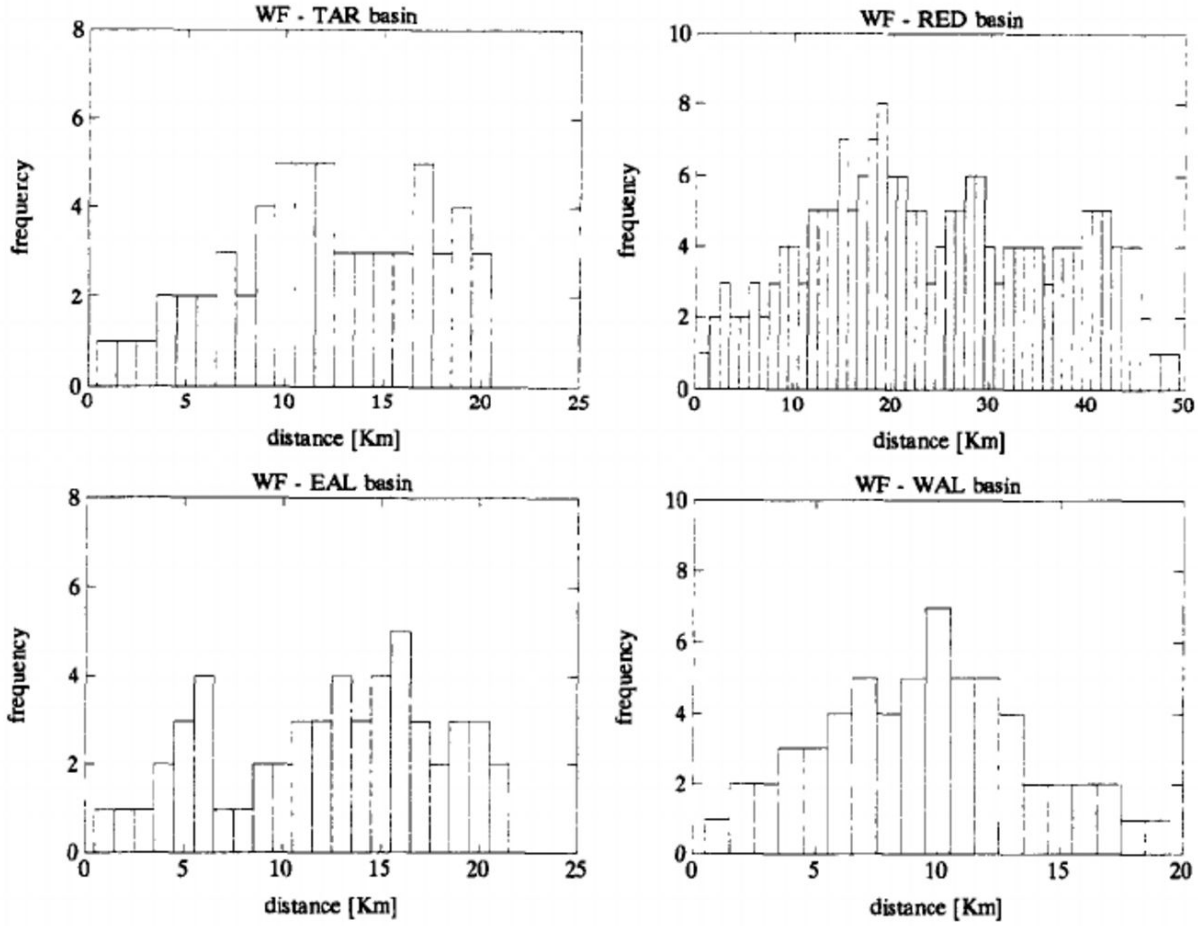

Fig. 4. Width functions of the four sub-basins.

GIUH that, for a given rainfall–runoff event, the streamflow velocity is approximately the same at any moment throughout the whole basin and that the 'reference velocity' can be taken as the velocity at the time of peak discharge for a given storm event (Rodriguez-Iturbe et al., 1979). As a consequence, the celerity c may be also considered constant over all the network. The diffusion coefficient D may also be considered as a reference average value for the hypothetical event: the assumption of

Table 2

Numerical values of the various quantities used to define the hydraulic conditions of the numerical example, calculated with reference to a wide rectangular section

Velocity	Discharge	Celerity	Diffusivity	Slope	Depth	Manning coefficient	Width
$v = \frac{1}{n} R^{2/3} S_o^{1/2}$	$Q = By_o v$	$c = kv$	$D = \frac{Q}{2BS_f} \approx \frac{Q}{2BS_o}$	S_o	y_o	n	B
$\approx \frac{1}{n} y_o^{2/3} S_o^{1/2}$		$k \approx 1.5$	$\approx \frac{vy_o}{2S_o} = \frac{cy_o}{3S_o}$				
(m s^{-1})	(m^3 s^{-1})	(m s^{-1})	(m^2 s^{-1})	(m m^{-1})	(m)	($m^{-1/3}$ s)	(m)
1.45	180	2.18	1800	0.001	2.5	0.040	50

R, Hydraulic radius.

constant diffusion based on reference hydraulic conditions is widely applied in the estimation of an IUH when this is derived from the solution of the convective diffusion equation (Todini, 1996; Naden, 1992). Finally, it should be noted that the realistic but certainly arbitrary geometry selected for the cross-section at the basin outlet is only used to evaluate the reference hydraulic conditions. These, and not the geometry, are assumed to be representative for the whole basin, in line with the previously mentioned research results of Leopold and Maddock (1953) and Pilgrim (1976, 1977).

On the basis of the above assumptions, and employing the Manning formula, the numerical values of the various quantities were calculated with reference to a wide rectangular section, and are shown in Table 2. A single hydraulic condition was considered for all four sub-basins to highlight the effect of the geomorphology on the IUH for a given hydraulic condition, for the cases of the WFIUH, the Horton GIUH and the actual GIUH.

On the basis of these reference hydraulic conditions and using the geomorphologic data previously described, three IUHs were calculated and are presented in Figs. 5 and 6. In Fig. 5, these are the WFIUH, the Horton GIUH (Eqs. (5) and (7)) and the GIUH based on the actual values of θ_i, p_{ij} (Eqs. (1) and (2)) but holding times based

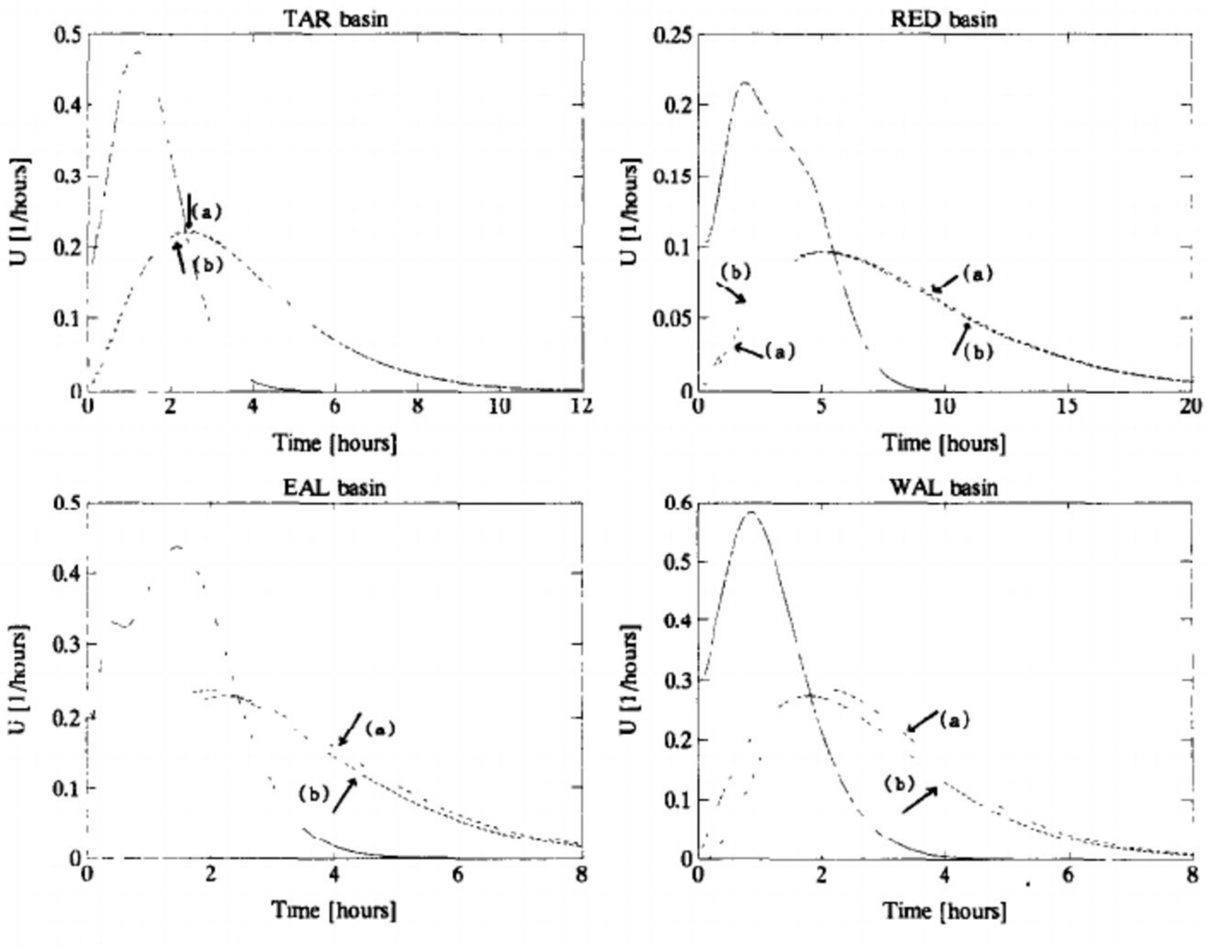

Fig. 5. Comparison between the WFIUH and the GIUH. Continuous line, WFIUH. Dashed–dotted line (a), Horton GIUH (parameters estimated through Eqs. (7a)–(7f)). Dashed line (b), GIUH with actual values for θ_i and p_{ij} (Eqs. (1) and (2)) but holding times based on the Horton length ratio (Eq. (7f)).

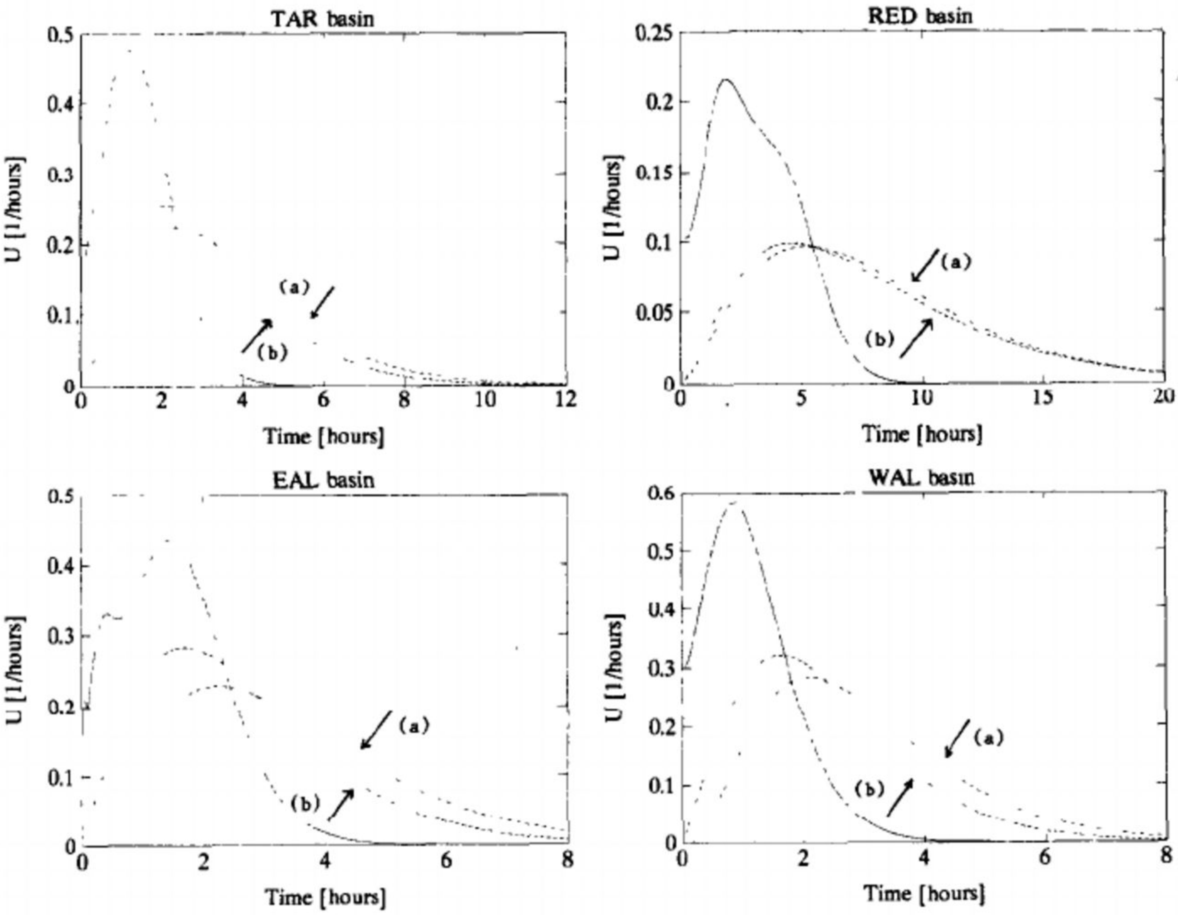

Fig. 6. Comparison between the WFIUH and the GIUH. Continuous line, WFIUH. Dashed–dotted line (a), Horton GIUH (parameters estimated through Eqs. (7a)–(7f)). Dashed line (b), actual GIUH (parameters estimated through Eqs. (1), (2) and (6)).

on the Horton length ratio (Eq. (7f)); in Fig. 6, these are the WFIUH, the Horton GIUH (Eqs. (5) and (7)) and the actual GIUH (Eqs. (1), (2), (5) and (6)).

In analysing these figures, two observations are appropriate. Referring to the GIUH, it is evident that the results are not significantly different if either the estimated or the actual values of θ_i and p_{ij} (see Fig. 5) are used. In contrast, Fig. 6 shows a significant difference when the actual average lengths of the streams of different order are used (see Eq. (6)) rather than those obtained through Eq. (7f). It is evident that the GIUH in not sensitive to the method used to calculate θ_i and p_{ij} and, therefore, to the morphologic information embedded in these quantities, given the clear differences between these two sets of values (see Table 1). However, the strong difference in the results obtained using the true average lengths instead of the Horton length ratios suggests that the actual information should be extracted from a map of a basin without filtering or smoothing it through the use of the Horton ratios.

However, what is more evident from an analysis of Figs. 5 and 6 is the great difference in the behaviour of the WFIUH and GIUH models. Obviously, the undulations in the WFIUH for the sub-basins RED and EAL are strictly related to the WF shape and cannot be expected in the GIUH because of the a priori assumptions which constrain its shape. However, there is a large difference in the time bases of the two schemes, even if they refer to the same geomorphologic information (although summarized in a different way) and the same hydraulic conditions. Given this discrepancy, a natural reaction of a practically minded hydrologist might be to estimate the 'time of concentration' of the network, T_c^{nw}, as the ratio of the longest distance from the outlet (see Fig. 4) to the velocity v and to compare it with the base time t_b of both IUHs. Working in this way one obtains

$$\begin{aligned} &\text{TAR basin}: \ L = 20 \ \text{km}; \quad T_c^{nw} \approx 5.0 \ [\text{h}] \\ &\text{Red basin}: \ L = 49 \ \text{km}; \quad T_c^{nw} \approx 10.0 \ [\text{h}] \\ &\text{EAL basin}: \ L = 21 \ \text{kml}; \quad T_c^{nw} \approx 5.0 \ [\text{h}] \\ &\text{WAL basin}: \ L = 19 \ \text{km}; \quad T_c^{nw} \approx 4.0 \ [\text{h}] \end{aligned} \tag{13}$$

where L is the longest flow distance in the basin network (see Fig. 4).

This pragmatic estimation of the concentration time, of course, disregards the delay effect owing to the storage capacity of the network and the consequent increase in its value. This damping effect is represented in the WFIUH through the coefficient D, and in the GIUH through the λ_i values, which, as already shown, can be readily interpreted as the time constants of linear reservoirs. It is therefore to be expected that the observed time base t_b will be longer than the estimated T_c^{nw}, and, on the basis of this, the values relevant to the WFIUH seem sensible (see Figs. 5 and 6), given also the fact that the diffusivity value $D = 1800 \ \text{m}^2\,\text{s}^{-1}$ is not very large. It is then surprising to see that the t_b of the GIUH is almost more than twice the t_b of the WFIUH for all the four cases considered here. Such large time base values cannot be explained by saying that the GIUH exhibits a larger geomorphologic dispersion than the WFIUH (see Rinaldo et al. (1991) for the definition of 'geomorphologic dispersion'). In fact, Snell and Sivapalan (1994) clearly showed that a geomorphologic IUH based on Strahler ordering exhibits an underlying dispersion of an order of magnitude less than a geomorphologic IUH based on an area and/or width function, hydraulic conditions being equal.

To find an explanation for such different behaviour, it is opportune to recall the reason for the presence of the velocity v in the GIUH model. This velocity has been introduced into the model with the assumption that the average travel time is proportional to the average length of the streams of given order. This fundamental assumption allows the estimation of all the coefficients λ_i through a single 'number' represented by the velocity v. However, as a travel time is certainly a length divided by a velocity, the question arises as to what is a 'distribution of lengths' in a Strahler ordering system and what is a 'velocity' in that framework? It is evident from the numerical application described here that v cannot be interpreted as the physical velocity of a drop of water coming from the furthest point of a basin, simply because the description of the underlying geomorphology of the natural drainage basin by means the Strahler ordering introduces a lumping action which, among other things, causes the loss of the information relevant to the maximum extent of the channel network itself, whereas, on the contrary, this information is maintained by the WF of a basin. Consequently, it may be opportune to abandon the idea of considering v as a physical velocity and to treat it as a pure calibration parameter or as a 'reference velocity'. In this context, a possible simple approach can be suggested based on a priori evaluation of the 'concentration time' T_c^{nw} of the channel network.

4.1. An estimation criterion for the parameter v of the GIUH

The procedure adopted to include the concentration time in the evaluation of the parameter v of the GIUH is as follows. Henderson (1963) observed that the most important characteristics of an IUH are the peak q_p and the time to peak t_p, and that, as long as these two factors are correct, the exact form of an IUH is not very important. Thus, Rodriguez-Iturbe and Valdés (1979) resorted to an accurate approximation of the GIUH, involving values of q_p and t_p obtained from the full expression of the GIUH itself, for different velocities in the range 0.5–6 m s^{-1} and for different basin orders $\Omega = 3, 4, 5$, with $\overline{L}_1$ varying from 125 to 2000 m. The calculations were carried out for 126 combinations of values of R_A, R_B, and R_L in the ranges 2.5–5.0, 3.0–6.0 and 1.5–4.1, respectively. For fixed values of R_A, R_B, R_L, $\overline{L}_1$, Ω and v, the couples q_p, t_p were calculated and, finally, through a regression analysis, Rodriguez-Iturbe and Valdés (1979) obtained two relationships between q_p, t_p, and R_A, R_B, R_L, $\overline{L}_1$, Ω and v. Subsequently, Rosso (1984) rearranged these two equations in consistent units, to obtain

$$q_p = 0.364 R_L^{0.43} v L_\Omega^{-1} \quad (14a)$$

$$t_p = 1.584 (R_B/R_A)^{0.55} R_L^{-0.38} v^{-1} L_\Omega \quad (14b)$$

where L_Ω represents the length of the stream of the highest order. For a pure Hortonian basin the following equation holds: $L_\Omega = \overline{L}_1 R_L^{\Omega-1}$.

Then, in line with Henderson's observation (1963), these two equations could be used to define completely an IUH for an imposed shape. Nevertheless, for a given basin, even if the Horton ratios R_A, R_B, R_L and the length of the highest-order stream are directly derived from a map, the peak value q_p and the time to peak t_p, and then the IUH, are not fully defined by Eqs. (14a)–(14b) until the velocity v is selected. Thus, a further equation could be introduced to relate the velocity v to some characteristic of the IUH that can be a priori estimated. The base length or time base t_b of the instantaneous unit hydrograph may be used for this purpose; in fact, it can be also interpreted as the time to concentration of the channel network (see, e.g. Henderson, 1963; Bras, 1990), which, in turn, can be easily defined in many practical applications to real-world basins, even if in a rough way. For example, this can be done by using some empirical formulae which are usually fitted to the basins of a particular country or a particular area of it. Furthermore, in the case that a basin is under the management of a Public Agency or Centre charged with flood prediction–prevention or the collection of hydrologic data, then it is expected that any agency hydrologist can evaluate the range of the concentration time of that channel network on the basis of his or her experience.

To define, in an easy way, the third equation that relates the velocity v, or possibly the ratio $v^{-1}L_{\Omega}$, to the concentration time of the channel network, we consider the GIUH approximation developed by Rosso (1984). Starting from Eqs. (14a) and (14b), Rosso (1984) derived the two parameters of the Nash model (1957):

$$f(t;\alpha,k) = [k\Gamma(\alpha)]^{-1}(t/k)^{\alpha-1}\exp(-t/k) \tag{15}$$

as

$$\alpha = 3.29(R_B/R_A)^{0.78}R_L^{0.07} \tag{16a}$$

$$k = 0.70[R_A/(R_BR_L)]^{0.48}v^{-1}L_{\Omega} \tag{16b}$$

The gamma function (Nash model) parametrized through equations (16a) and (16b) (subsequently referred to as ΓGIUH) and the GIUH are basically equivalent (Rosso, 1984; Chutha and Dooge, 1990), given the same Horton ratios and reference velocity. It is then possible to work with Eqs. (15), (16a) and (16b) instead of working on the full formulation of the GIUH, which, in the case of higher-order basins, becomes somewhat intractable.

Eq. (16b) contains the ratio $v^{-1}L_{\Omega}$. This ratio can be easily related to the concentration time T_c^{nw} by equating it to the time base t_b of the ΓGIUH, now defined as follows:

$$\int_0^{t_b}\mathrm{GIUH}(\tau)\mathrm{d}\tau = 0.99$$

Then, it is possible to generate many triplets R_B, R_L and R_A, subjected to the necessary constraints (see Appendix), and for any value of the ratio $v^{-1}L_{\Omega}$, ranging between 0.1 and 15 h, to calculate the corresponding t_b value. Finally, through a simple regression analysis, one obtains

$$\frac{L_{\Omega}}{v} = 0.138T_c^{nw}(R_B/R_A)^{0.003}R_L^{0.437} \quad \text{(determination coefficient} = 0.992) \tag{17}$$

(where T_c^{nw} is in h, v is m h^{-1} and L_{Ω} is in m). It should be noted also that the ratio R_B/R_A can be disregarded; this result seems reasonable because a 'travel time' is certainly linked to length information and not to area and/or bifurcation information.

Then, Eq. (17) may be written approximately as

$$\frac{L_{\Omega}}{v} = 0.138T_c^{nw}R_L^{0.437} \tag{18}$$

or

$$v = L_{\Omega}(0.138T_c^{nw}R_L^{0.437})^{-1} \tag{19}$$

and, from Eq. (18)

$$T_c^{nw} \propto v^{-1}L_{\Omega}R_L^{-0.437}$$

Comparing this last equation with the "controlling parameter I in hydrologic similarity for basins under the same kinematic conditions" (Rodriguez-Iturbe et al., Eq. (13), 1979) $I = R_L^{0.43}/L_\Omega$, one may conclude that the hydrologic similarity parameter is nothing more than an indirect evaluation of the basin channel network time of concentration.

Going back to the numerical example related to the four sub-basins, the T_c^{nw} values can be estimated for the four sub-basins through Eqs. (13), i.e. disregarding any diffusion effect, and then, through Eq. (19), the following values of velocity are obtained:

$$\begin{aligned} &\text{TAR basin}: \quad v = 3.5\ (\mathrm{m\,s^{-1}}); \quad L = 20\ \mathrm{km} \\ &\text{RED basin}: \quad v = 4.4\ (\mathrm{m\,s^{-1}}); \quad L = 49\ \mathrm{km} \\ &\text{EAL basin}: \quad v = 3.2\ (\mathrm{m\,s^{-1}}); \quad L = 21\ \mathrm{km} \\ &\text{WAL basin}: \quad v = 3.4\ (\mathrm{m\,s^{-1}}); \quad L = 19\ \mathrm{km} \end{aligned} \tag{20}$$

where L is largest distance from the outlet.

It should be noted that for all these velocities the ratio L/v is definitely different

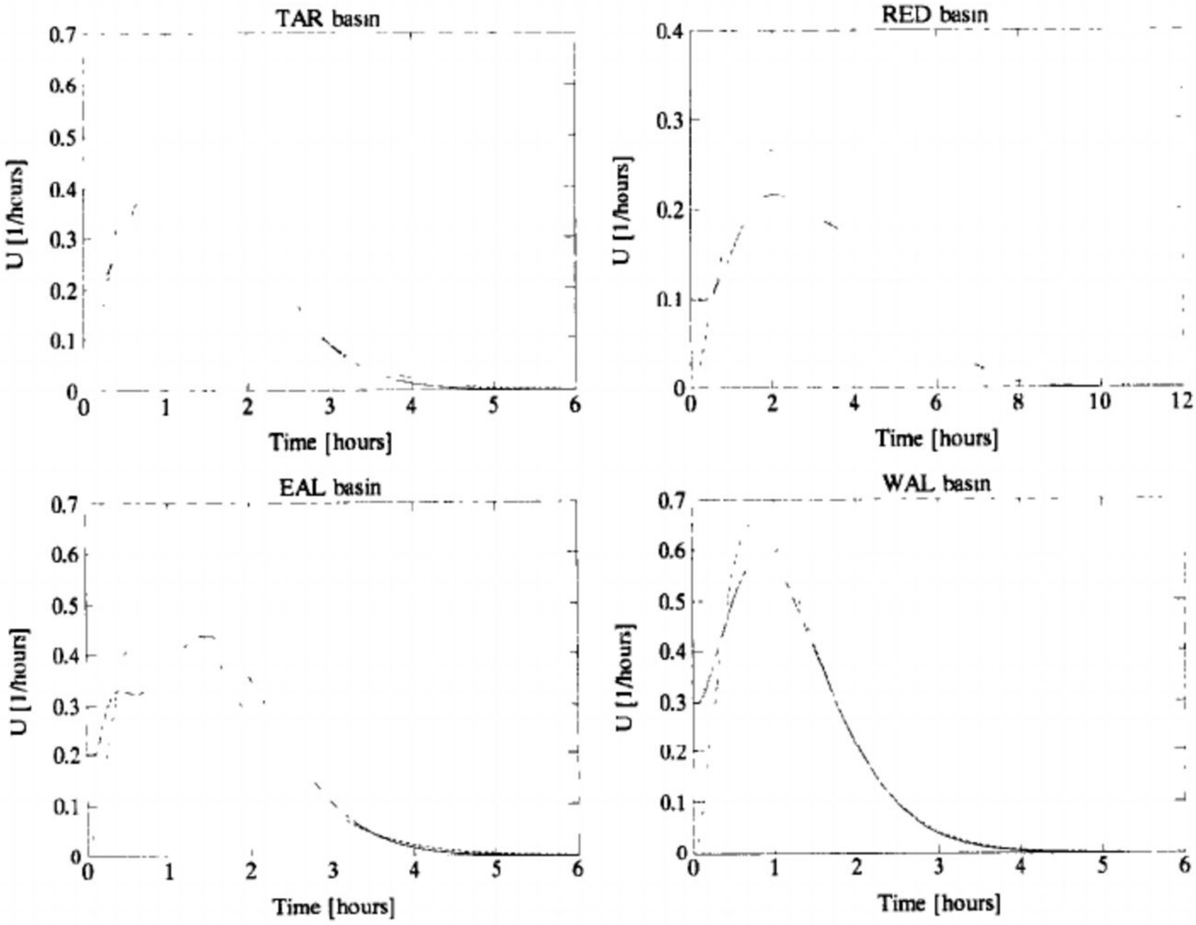

Fig. 7. Comparison between the WFIUH and the GIUH. Continuous line, WFIUH; dashed line, GIUH represented through the gamma approximation (ΓGIUH, see Eq. (15)). The reference velocity values of the GIUHs are calculated according to Eq. (19).

from the estimated concentration time T_c^{nw} (see Eqs. (13)). Therefore, none of these velocities can be considered as the physical velocity of the drop of water along the river network of the basins.

Using these new values for the reference velocity of the GIUH model, the GIUHs have been re-calculated, and are presented in Fig. 7 together with the previously obtained WFIUHs: the GIUHs are approximated here through the ΓGIUH (Eqs. (15) and (16)) and the reference velocity is calculated by using Eq. (19) (see Table 1 for the values used); they are now more similar to the corresponding WFIUHs. In Fig. 8, similar plots are presented, but in this case the actual GIUHs (Eqs. (1), (2) and (6)) are shown. Again, by comparing Fig. 7 with Fig. 8, it is possible to highlight the different results obtained by using the Horton ratios and the actual values of θ_i, p_{ij} and $\overline{L}_i$, respectively.

Furthermore, whatever the method used to estimate p_{ij} and $\overline{L}_i$, it is still possible to observe some interesting differences between the WFIUHs and the GIUHs. When the WF has a marked negative skewness (see, e.g. the WF of the TAR and the EAL basins) the GIUH has a smaller time to peak than that of the WFIUH. This does not happen for the WAL basin, whose WF is almost symmetric. This suggests that the lumping approach inherent in the nature of the GIUH tends to concentrate all the

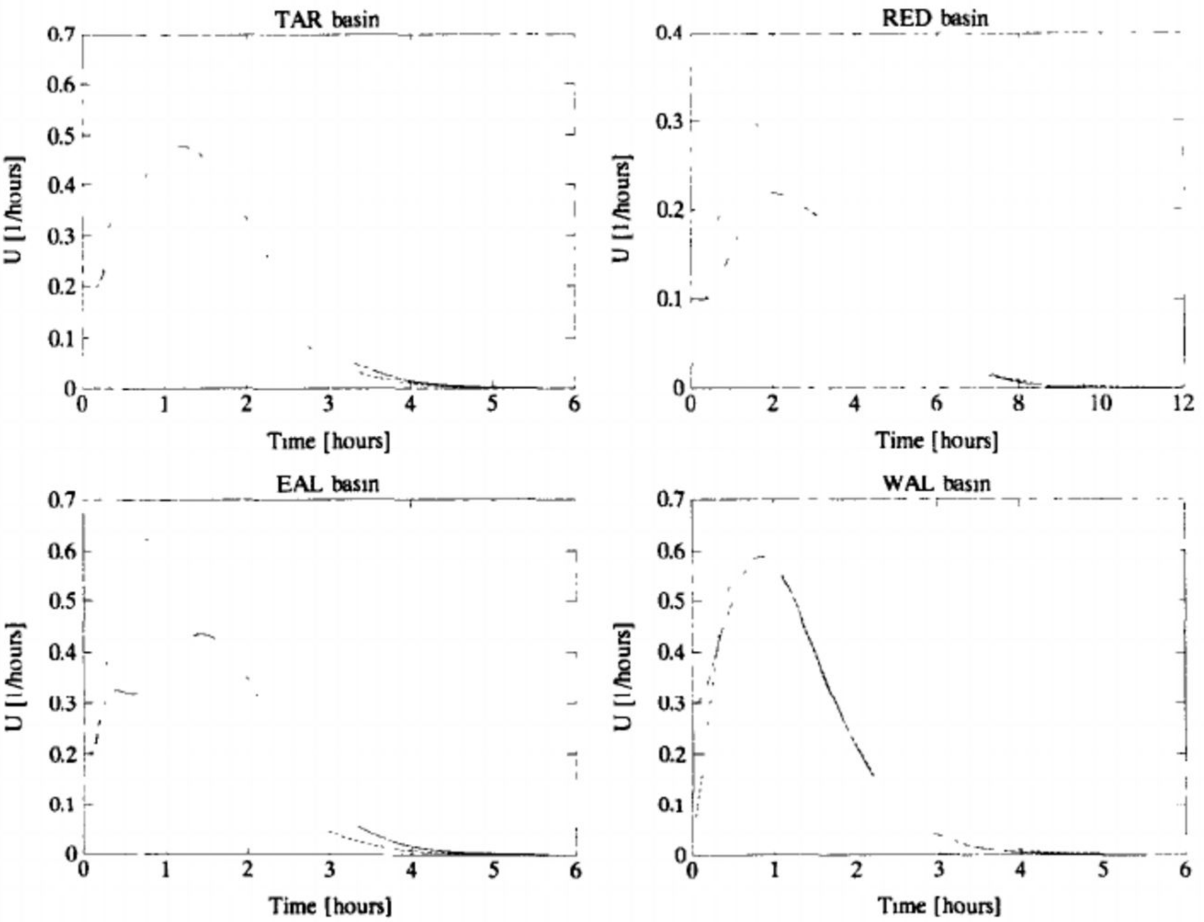

Fig. 8. Comparison between the WFIUH and the GIUH. Continuous line, WFIUH; dashed line, actual GIUH (parameters estimated through Eqs. (1), (2) and (6)).

morphologic information around the basin centroid. In other words, by describing the characteristics of the network with a concise geomorphologic parametrization, information about the network shape (easily obtained from maps) may be lost.

5. Conclusions

The GIUH is based on classifying the channel network of a basin according to Strahler's ordering scheme, and the geomorphologic information embedded in the model is represented by the probability of state i, θ_i, and by the transition probability from a given state i to some state of higher order j, p_{ij}. The probability density function of the time spent by a drop of water in a state of order i is assumed to be of the exponential type; then, the parameter λ_i of that distribution represents the reciprocal of the mean holding time in any stream of order i. No holding time is considered for the overland flow. The assumption of an exponential distribution for the holding time makes the GIUH mathematically equivalent to the response of a conceptual model consisting of linear storage elements in parallel and in series.

The GIUH is not very sensitive to the estimation of the parameters θ_i and p_{ij}, and so the estimates based on the Horton ratios or the actual values extracted directly from a map might be used. However, the GIUH is extremely sensitive to the estimation of $\overline{L}_i$ and, in turn, of λ_i. In this case, the use of the information extracted directly from the map should be preferred.

The reference velocity v to be used in the GIUH cannot be physically interpreted as the velocity of a drop of water coming from the farthest point in a basin. The work presented here suggests that v be considered purely as a calibration parameter which can be estimated in practice through a relationship with the concentration time, the Horton length ratio and the length of the stream of the highest order of the channel network.

The controlling parameter I (Rodriguez-Iturbe et al., 1979, Eq. (13)) in hydrologic similarity, for basins under the same kinematic conditions, has been shown here to be an indirect evaluation of the basin channel network time of concentration.

In general, by summarizing the characteristics of the network in terms of the geomorphologic parameters on which the GIUH is based, information about the shape of the network may be lost. The WF approach allows more information about the network to be retained and it is straightforward to implement by simply using a map of the basin; in fact, it does not need the estimate of any morphologic or topologic parameter. Furthermore, it can be easily coupled with any routing scheme, possibly of the hydraulic type such as the convective diffusion equation, so that the values of the parameters can be defined in a sensible way, in accordance with the hydraulic characteristics of the network.

Acknowledgements

This study has been undertaken within the framework of a continuing programme of collaboration between the University of Bologna and the University of Newcastle upon Tyne. The authors wish to thank Dr. Chris Kilsby for his help in calculating the morphologic paramcters of the basins used in this study, and thank all the reviewers for their helpful comments on the first draft of the paper.

Appendix. Strahler ordering scheme and the Horton ratios

The Strahler ordering scheme (1952) is based on the following statements. The furthest upstream points in the network are termed sources and the point furthest downstream the outlet. A junction is termed the point where two streams (or links, or river segments) combine to form a single stream. Streams that originate at a source are defined as first-order streams. When two streams of the same order join, the stream that is formed has an order one degree higher than the common order of the two streams which combine. When two streams of different orders join, the combined stream has the higher order of the two combining streams. The order of the basin is the highest stream order Ω.

Horton's laws (1945) may be summarized as follows:

$$\text{law of stream numbers}: \frac{N_\omega}{N_{\omega+1}} = R_B$$

$$\text{law of stream lengths}: \frac{\overline{L}_\omega}{\overline{L}_{\omega+1}} = R_L$$

$$\text{law of stream areas}: \frac{\overline{A}_\omega}{\overline{A}_{\omega+1}} = R_A$$

where N_ω is the number of streams of order ω, $\overline{L}_\omega$ is the mean length of streams of order ω, and $\overline{A}_\omega$ is the mean area of the basins of order ω. R_B, R_L and R_A represent the bifurcation ratio, the length ratio and the area ratio, respectively, whose values in nature are normally between three and five for R_B, between 1.5 and 3.5 for R_L and between three and six for R_A. Furthermore, $R_B \leqslant R_L \leqslant R_A$.

References

Abramowitz, M. and Stegun, I.A., 1965. Handbook of Mathematical Functions. Dover, New York.

Baldwin, E.P. and Potter, K.W., 1987. Improving flood quantile estimation on ungauged watersheds. In: V.P. Singh (Editor), Regional Flood Frequency Analysis. D. Reidel, Dordrecht, pp. 65–75.

Bras, R.L., 1990. Hydrology: an Introduction to Hydrologic Science. Addison Wesley, Reading, MA.

Chutha, P. and Dooge, J.C.I., 1990. The shape parameters of the geomorphologic unit hydrograph. J. Hydrol., 117: 81–97.

Dooge, J.C.I., 1959. A general theory of the unit hydrograph. J. Geophys. Res., 64(2): 241–256.

Dooge, J.C.I., 1973. Linear theory of hydrologic systems. US Dep. Agric. Agric. Res. Serv. Tech. Bull., 1468.

Gupta, V.K., Waymire, E. and Wang, C.T., 1980. Representation of an instantaneous unit hydrograph from geomorphology. Water Resour. Res., 16(5): 855–862.

Henderson, F.M., 1963. Some properties of the unit hydrograph. J. Geophys. Res., 68(10): 4785–4793.

Horton, R.E., 1945. Erosional development of streams and their drainage basins: hydro-physic approach to quantitative geomorphology. Geol. Soc. Am. Bull., 56: 275–370.

Howard, R.A., 1971. Dynamic Probabilistic Systems. Wiley, New York.

Karlinger, M.R. and Troutman, B.M., 1985. Assessment of the instantaneous unit hydrograph derived from the theory of topologically random networks. Water Resour. Res., 21(11): 1693–1702.

Kirkby, M.J., 1976. Tests of the random network model and its application to basin hydrology. Earth Surface Processes, 1: 197–212.

Leopold, L.B. and Maddock, T. Jr., 1953. The hydraulic geometry of stream channels and some geomorphologic implications. US Geol. Surv. Prof. Pap., 252, 56 pp.

Mesa, O.J. and Mifflin, E.R., 1986. On the relative role of hillslope and network geometry in hydrologic response. In: V. Gupta, I. Rodriguez-Iturbe and E. Wood (Editors), Scale Problems in Hydrology. D. Reidel, Dordrecht, pp. 1–17.

Naden, P.S., 1992. Spatial variability in flood estimation for large catchments: the exploitation of channel network structure. J. Hydrol. Sci., 37(1–2): 53–71.

Nash, J.E., 1957. The form of the instantaneous unit hydrograph. IASH Publ., 42: 114–118.

Pilgrim, D.H., 1976. Travel times and nonlinearity of flood runoff from a tracer study on a small watershed. Water Resour. Res., 12(4): 487–496.

Pilgrim, D.H., 1977. Isochrones of travel time and distribution of flood storage from a tracer study on a small watershed. Water Resour. Res., 13(3): 587–595.

Rinaldo, A., Marani, A. and Rigon, R., 1991. Geomorphological dispersion. Water Resour. Res., 27(4): 513–525.

Rodriguez-Iturbe, I. and Valdès, J.B., 1979. The geomorphologic structure of hydrologic response. Water Resour. Res., 15(6): 1409–1420.

Rodriguez-Iturbe, I., Devoto, G. and Valdés, J.B., 1979. Discharge response analysis and hydrologic similarity: the interrelation between the geomorphologic IUH and the storm characteristics. Water Resour. Res., 15(6): 1435–1444.

Rosso, R., 1984. Nash Model relation to Horton order ratios. Water Resour. Res., 20(7): 914–920.

Shreve, R.L. 1966. Statistical law of stream number. J. Geol., 74: 17–37.

Snell, J.D. and Sivapalan, M., 1994. On geomorphological dispersion in natural catchments and the geomorphological unit hydrograph. Water Resour. Res., 30(7): 2311–2323.

Strahler, A.N., 1952. Hypsometric (area–altitude) analysis of erosional topography. Geol. Soc. Bull., 69: 1117–1142.

Todini, E., 1996. The ARNO rainfall–runoff model. J. Hydrol., 175: 339–382.

Troutman, B.M. and Karlinger, M.R., 1984. On the expected width function for topologically random channel networks. J. Appl. Prob., 21: 836–884.

Troutman, B.M. and Karlinger, M.R., 1985. Unit hydrograph approximation assuming linear flow through topologically random channel networks. Water Resour. Res., 21(5): 743–754.

Troutman, B.M. and Karlinger, M.R., 1986. Averaging properties of channel networks using methods in stochastic branching theory. In: V. Gupta, I. Rodriguez-Iturbe and E. Wood (Editors), Scale Problems in Hydrology. D. Reidel, Dordrecht, pp. 185–216.